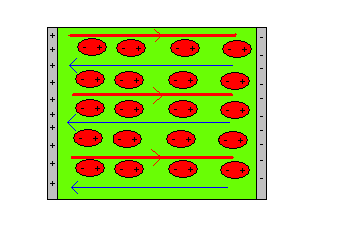
**Energy stored in the Fields**

**Energy Stored in Electric Field within an Insulator**

The energy stored within a charge configuration is the work required to set it up. We already ascertained how much energy is stored in a charge configuration in a vaccum. Let’s consider a charge distribution in the presence of a dielectric. So we presume the dielectric is initially unpolarized – so no bound charge – and, concommitantly, we have no free charge distribution set up as of yet. But then we begin to introduce free charge in some region of space, ρf, and as we do so, the bound charge ρb, will start to develop as the insulator becomes polarized. Archetypal situation is capacitor filled with a dielectric – for visualization’s sake.



So let’s say we have a material with free and bound charge in it. Let’s determine the minimum amount of work required to add more free charge to the distribution. This would be:



Now as we add up charges we will be incrementing **D** from its starting value of 0 to its final value of **D**. To get the total work we would say:



But we cannot integrate over δD unless we know how **E** varies with **D**. So here we will use our linearity model and say **E** = **D**/ε. Then we have:



which can be written as:



(also Usys) We can do this another way that starts from a work calculation more directly. So consider a short time interval dt. Then the instantaneous power exerted (by us, which is negative the power exerted by field) on the free charges during this time interval would be:



Now plug in ∇·**D** = ρf. So,



Could write this therefore as:



So now we basically have what we had found earlier. If we now assume that we have a linear medium, whereby **P** = χε0**E**, and hence **D** = ε0**E** + **P** = ε0(χ+1)**E** = ε**E**, then we can write this as:



Integrating from t = ti to tf, we can conclude,



**Work we do on the Insulator as we set up the Electric Field**

This is a useful thing to calculate, especially apropos Thermodynamics, so here we go. So when we move free charges around, their field exerts forces on the bound charges. And the work it does in some infintesimal time interval would just be the power it exerts during that time. And this is:



But now we have ρb = -∇·**P**. So,



Or something like that. And considering just a small change in time, we could say:



where **P** is the polarization. We can replace Ef = D/ε0 if we have the requisite geometry, (like capacitor plate geometry, and linear homogeneous substances, etc.) Note this work term does *not* include the work required to set up the free electric field itself, i.e., (ε0/2)∫E2dτ.

**Other ways to write the Energy**

Given this, we can write the energy a few other ways. Going back to,



And fill in **D** = ε0**E** + **P**, is, and come to:



So now could say,



The first term is the rate of change of bulk electric field energy. The second term is the rate at which this field is doing work on the polaron. We can interpret this equation as saying that δUE (energy we put into this setup) is the work required to both create the bulk interstitial field and to enable that field to do work on the polaron (to stretch it). If we compare this to the formula we get when we apply the 1st Law of Thermodynamics, we’ll see that the last term in the box must be equal to the total change in potential energy, NPEatomic spring = NW1, of the each dipoles as they’re stretched. I guess this is so because our model for the dielectric has that the internal spring force between the dipoles is at all times equal to the force applied by the bulk interstitial field in equilibrium. And so the work done by the bulk interstitial field is the same as the work done by the internal spring force. *Another* way to write it is to say,



*If* the medium is linear, so that D and P are proportional to each other, then this reduces to:



This is basically the energy stored in the free field, minus the work the free field does on the polaron.

Now let’s broaden our perspective by placing our analysis in the context of the 1st Law of Thermodynamics. We have a system comprising all particles – both free and bound. The only work external to the system is the work required to assemble the free charges. From the first law of the thermodynamics, since there is no heat added, presumably, this work W = ΔU must be equal to the change in energy of the system. So:



It’d be nice to account for the energy of the system directly. It should be something like,



where f refers to free charges, and b to bound charges (this includes the +/- charges within each molecule). KEf, KEb refer to the kinetic energies of the free and bound particles. PEff is the electric potential energy between free particles, PEbb is the electric potential energy between bound particles, and PEf,b is that between free and bound particles. So the total PE = PEf,f + PEb,b + PEf,b. Now this is, where i and j run over all particles, free and bound:



Now apropos the f-f interaction in the top line, we could make the continuum approximation, as this ought to do the interaction justice, but then we have to be careful that we don’t include self-energy terms. But since the charges are spread out in a 2D structure, the self-energy will be zero in the continuum approximation, as argued in that energy file previous. There is no risk in going to the continuum approximation with the last set of terms, since there is no risk of self-energy terms here anyway. For the middle terms, we can go to continuum, but this will screw up the intramolecular field (the field between the +/- pair within the dipoles) that we need to account for. So we will implicitly go to the continuum approximation, simultaneously extend the sums to include self-terms, which should vanish because our charge distributions are 2D and 3D. But we will have to explicitly pull out the terms corresponding to intramolecular pairs because our continuum and polar approximation, as stated above, won’t work on these terms. So we have:



Combining the top three terms, we now have:



Of course the first guy is just the net bulk interstitial field, outside the dipoles. This is what we’ve been calling **E**, but again, it is only the **E** outside the dipoles, where our dipole approximation is valid. So now we have:



So we can write:



We can recognize the Σ∫dτ **E**b+·**E**b- term as just the sum of the internal potential energies of each of the dipoles. And KEb is of course their (could say internal) kinetic energy. So we could subsume those two terms together,



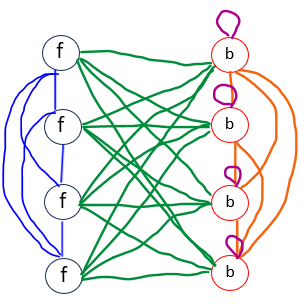
and just write:



We usually consider the minimum necessary work to assemble the free charges so that KEf = 0. And we also do it slowly/isentropically so that KEb = 0. So all we’re left with is the potential energy.



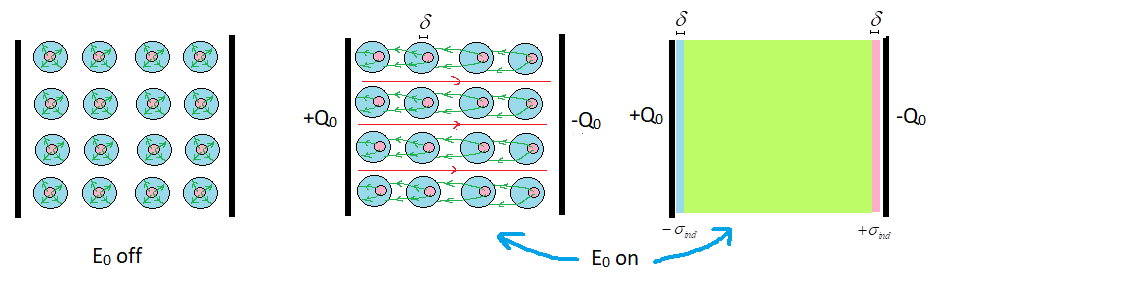
And this ΔUsys is precisely ΔU from above. So we have that the change in energy is the total change in potential energy of the system. The second term comprises all terms in the change in potential energy between all charges, free and bound, excepting only the term that accounts for the potential energy change between *just* the two individual charges in each dipole. *That* is what the *first* term accounts for. So the second term can be said to account for the *bulk* change in potential energy between free+free, free+bound, bound+bound. The only thing bulk change in potential energy doesn’t account for is the microscopic bound+bound potential energy, i.e., the potential energy between charges within the same dipole. Again, this is what the first term takes care off. One more time, I have a drawing below. The blue terms are PEff green terms are PEfb, and orange terms are PEbb. This is what (ε0/2)∫E2dτ takes care of. The purple terms are PEb, and this what ΔPEdipoles takes care of.



Noting where we started from, ΔUsys is often written as (ε0/2)∫dτ E2, which is technically true. But in that case, E would be the *total* field, which includes the bulk interstitial field (same as the E in the boxed formula above), and the field within the dipoles themselves. It’s this latter contribution that gives us the ΔPEdipoles term, basically.

**Model Dielectric Energy Calculations**

Just want to try to propose a simple classical, non-statistical (meaning we’ll presume it’s in just one state and so has only one response), model of the susceptibilities. Consider a dielectric placed between two plates again. We can model the atoms in the material as a positive nucleus, charge q, centered about a spherical electronic cloud of charge q.



For the sake of discussion we’ll assume there are n atoms per unit volume. The electric field between the plates, due to the charge Q0/-Q0 on each side, will polarize the molecules, displacing the electronic cloud a distance δ. From our previous analysis, we found this results in a dipole moment per unit volume of:



We also found that this resulted in an effective bound surface charge density of:



where χ is called the electric susceptibility. And also found the total field to be:



where χ is called the electric susceptibility.

**Energy as work done to assemble free charges (as dielectric gradually polarizes)**

What is the total potential energy stored? Well this would be the total work required to set up all charges, and so would be: ΔPE = ΔPEf,f + ΔPEb,b + ΔPEf,b, where the last term includes the interaction between the free and bound charges. What is the work? Well,



Now recall



and so,



And this must be the change in PE.

**Energy as bulk/interstitial electric field energy + work that field does on dipoles**

Here’s another way to do it, where we treat the energy as coming from the bulk field, and then also the energy needed to stretch the atomic ‘spring’ to separate the nucleus and cloud. This follows from the first alternate boxed formula for the energy. Would set it up like this (might want to review that Dielectric file to remember definitions of some of these terms):



So there we are.

**Energy as free field energy minus work it does on dielectric**

We can think about this a different way. So this approach basically follows from the last alternative formula for the energy. It relies on having a linear substance. First let’s break the energy formula down to:



where we use the formula for PEfb and also recognize that it’s just the (-) the work the free does on the bound (for linear substance), which must be equal to the change in PE of the bound guys. And so by work-energy equation, these must cancel. That leaves us with final result. So now we just have to evaluate:



So I don’t really know what to make of this. It doesn’t seem to me that on general physical grounds the total potential energy ‘should’ be equal to the free potential energy, minus the work it does on the bound charges. It *does* seem to me that it should be equal to the work we do to set up the free charges alone (this would be the free potential energy) + the work required to push the dielectric into the capacitor. So it would seem that the extra term must be this extra work: Wfree|bound = -Wus|bound. Well this would make Wus|bound negative. And this makes sense because the capacitor should suck the dielectric into itself. And so it is clear that for the same external charge Q0, the energy stored is less with the dielectric. And this makes sense since (?) the external field would have to do work to stretch the dipoles into position. On the other hand, for a given field/potential difference, the energy stored is greater. This also makes sense (?) since it would take more energy to set up such a field (with the opposing dipole field) than it would w/o the interfering dipoles. What is the work done by the external field (i.e. the free charges) on the bound charges, to engender their separation? For conceptual, ease, you can think of it like this: you keep charges stationary, increment the field, and then let the charges go. Then calculate the work done on them as they equilibrate to their new positions.



We’ll note that E is reduced, and so energy is reduced. This can be thought of in two ways. One: you now have a bunch of +/- dipoles inside the dielectric, and this is a negative potential energy situation so overall U ought to be reduced. On other hand, can think of work-energy equation. When you insert the capacitor, the field is actually going to draw it in (because it is a lower energy situation; can also see this with the field lines), and so the work the capacitor/field does to do this reduces its energy.

**Energy as change in internal potential energy of the system**

So we’re claiming W = ΔU = ΔUsys. This is practically virtually identical to the approach taken two sections up. So starting with:



So there we go! So for constant E, we get more energy because of the addition of those intramolecular potential energy terms, sometimes called the energy of stretching the bonds. Could also write it as:



Now the last term (change in) is just the work required to move the clouds from center to off-center by δ. Or move the nucleus from center to off-center. And for one of these molecules, this amounts to:



Now there are N such molecular terms, so,



Putting this back in, we have all total,



So there we go, again.